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R and R_τ Ratios at the Five-Loop Level of Perturbative QCD

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We study perturbative QCD at the five-loop level. In particular we consider $R = \sigma_{\mathrm{tot}}(e^+e^- \to \mathrm{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ and $R_\tau = \Gamma(\tau \to \nu + \mathrm{hadrons})/\Gamma(\tau \to e\nu\bar{\nu})$. We use our method to estimate the five-loop coefficients. As a result, we obtain $\alpha_s(M_Z) = 0.1186(11)$ and $\alpha_s(34\ \mathrm{GeV}) = 0.1396(16)$, which are accurate at the 1% level. We also find $R = 3.8350(18)$, which is consistent with R_τ and is accurate to 0.05%.

Perturbative QCD has been used to describe the strong interaction very successfully, when the energy scale is large enough. This includes the R ratio

$$R = \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{1}$$

and also the R_τ ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu + \mathrm{hadrons})}{\Gamma(\tau \to e\nu\bar{\nu})} \tag{2}$$

even though the mass scale M_τ is not very large. Recently Braaten (1993) presented a discussion of R_τ and a new quantity as well, the spin asymmetry parameter

$$A_\tau = \frac{R_F - R_B}{R_F + R_B} \tag{3}$$

R_F and R_B are the "forward" and "backward" components of R_τ:

$$R_\tau = R_F + R_B \tag{4}$$

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The lowest order estimates are

$$R_\tau = 3$$

and

$$A_\tau = 1/3 \tag{5}$$

These estimates are changed by perturbative and nonperturbative corrections as follows:

$$R_{\rm F} = 2S_{\rm EW}(1 + f_{\rm F}^{(0)} + f_{\rm F}^{(2)} + f_{\rm F}^{(4)} + f_{\rm F}^{(6)} + \cdots)$$

and

$$R_{\rm B} = S_{\rm EW}(1 + f_{\rm B}^{(0)} + f_{\rm B}^{(2)} + f_{\rm B}^{(4)} + f_{\rm B}^{(6)} + \cdots) \tag{6}$$

where

$$S_{\rm EW} = 1.019 \tag{7}$$

is the electroweak correction and the $f_{\rm F}^{(n)}$ and $f_{\rm B}^{(n)}$ are proportional to $1/M_\tau^n$ with coefficients that depend logarithmically on M_τ.

The purely perturbative QCD effects from the interactions of massless quarks and gluons for $N_f = 3$ are

$$f_{\rm F}^{(0)} = \frac{\alpha_s}{\pi} + 5.765\left(\frac{\alpha_s}{\pi}\right)^2 + 34.48\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(3)} + 165.1)\left(\frac{\alpha_s}{\pi}\right)^4 \tag{8}$$

and

$$f_{\rm B}^{(0)} = \frac{\alpha_s}{\pi} + 4.077\left(\frac{\alpha_s}{\pi}\right)^2 + 10.12\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(3)} - 96.1)\left(\frac{\alpha_s}{\pi}\right)^4 \tag{9}$$

where $\alpha_s = \alpha_s(M_\tau)$ is the running coupling constant of QCD in the $\overline{\rm MS}$ scheme evaluated at the scale M_τ. The coefficient $d_4^{(3)}$ is the fifth coefficient in the series

$$d_0 = 1, \qquad d_1 = 1, \qquad d_2 = 1.64, \qquad d_3 = 6.37$$

and has not yet been calculated [perturbative expansion of $-2\pi^2 s(d/ds)\pi^{(1)}(s)$]. We will use our estimation method which makes use of Padé approximants to estimate the value of d_4.

From equation (8) the Padé approximant prediction (PAP) is $d_4^{(3)} = 41$. From it equation (9) is $d_4^{(3)} = 116$. Applying it directly to the d_i series we obtain $d_4^{(3)} = 31$. The average is $d_4^{(3)} = 55$. Finally the PAP for the $(\alpha_s/\pi)^4$ term for R_τ is 133. Thus $d_4^{(3)} = 133 - 78 = 55$, in agreement with the average above. For further details, see our earlier papers (Samuel and Li, 1992; Samuel *et al.*, 1993*a*,*b*, 1994). Thus we take as our value, with conservative error estimates,

$$d_4^{(3)} = 55^{+60}_{-24} \tag{10}$$

This is our result for $N_f = 3$.

Our results for

$$f_F^1 = f_F^{(2)} + f_F^{(4)} + f_F^{(6)} = 0.0304$$

and

$$f_B^1 = f_B^{(2)} + f_B^{(4)} + f_B^{(6)} = -0.1082 \tag{11}$$

agree with those of Braaten. The relative contribution to R_τ is

$$\frac{2f_F^1 + f_B^1}{3} = -1.58\% \tag{12}$$

There are various experimental values for R_τ. We use the world average (Davier, 1993) for B_e and B_μ. From $B_e = 17.76(15)\%$ we obtain

$$R_\tau = 3.658(31) \tag{13}$$

and from $B_\mu = 17.53(19)\%$ we obtain

$$R_\tau = 3.629(24) \tag{14}$$

The weighted average of (13) and (14) is

$$R_\tau = 3.640(19) \tag{15}$$

From the measured τ lifetime (Trischuk, 1993; Marciano, 1992) $\tau_\tau = 0.2957(32) \times 10^{-12}$ sec one obtains

$$R_\tau = 3.549(39) \tag{16}$$

We take as our value the weighted average of (15) and (16)

$$R_\tau = 3.623(17) \tag{17}$$

(see also Samuel, 1993).

Now from equations (4), (6)–(10), (12), and (17) we obtain our result for $\alpha_s(M_\tau)$,

$$\alpha_s(M_\tau) = 0.3233(89) \tag{18}$$

and using equation (3) we obtain

$$A_\tau = 0.413(22) \tag{19}$$

Equation (19) agrees with Braaten's result. Our result for $\alpha_s(M_\tau)$ is somewhat different from Braaten's,

$$\alpha_s(M_\tau) = 0.319(17) \tag{20}$$

Note, however, that the error in equation (18) is much smaller than the error in equation (20). Actually, due to an interpolating error, Braaten's result should be

$$\alpha_s(M_\tau) = 0.324(17) \tag{21}$$

We can now obtain $\Lambda^{(3)}$ from the running of α_s,

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 L}\left[1 - \frac{\beta_1 \ln 2L}{2\beta_0^2 L} + \frac{1}{4L^2\beta_0^4}(\beta_1^2 \ln^2 2L - \beta_1^2 \ln 2L + \beta_2\beta_0 - \beta_1^2)\right] \tag{22}$$

where $\beta_0, \beta_1, \beta_2$ are the coefficients of the QCD β function,

$$\begin{aligned} \beta_0 &= 11 - \frac{2N_f}{3} \\ \beta_1 &= 102 - \frac{38N_f}{3} \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2 \end{aligned} \tag{23}$$

$L = \ln \mu/\Lambda$ and N_f is the number of fermions (quarks). For $N_f = 3$ we obtain from equations (18), (22), and (23)

$$\Lambda^{(3)} = 352(17)\ \text{MeV} \tag{24}$$

We use the $\overline{\text{MS}}$ scheme throughout this paper.

In order to ensure continuity of α_s as N_f changes, we have derived the following relationships:

$$\begin{aligned} \Lambda^{(6)} &= \Lambda^{(5)}\left(\frac{\Lambda^{(5)}}{m_t}\right)^{2/21}\left[\ln\left(\frac{m_t^2}{\Lambda^{(5)2}}\right)\right]^{-321/3381}\left(\frac{23}{21}\right)^{13/49} \\ \Lambda^{(5)} &= \Lambda^{(6)}\left(\frac{m_t}{\Lambda^{(6)}}\right)^{2/23}\left[\ln\left(\frac{m_t^2}{\Lambda^{(6)2}}\right)\right]^{321/3703}\left(\frac{21}{23}\right)^{174/529} \\ \Lambda^{(5)} &= \Lambda^{(4)}\left(\frac{\Lambda^{(4)}}{m_b}\right)^{2/23}\left[\ln\left(\frac{m_b^2}{\Lambda^{(4)2}}\right)\right]^{-963/13225}\left(\frac{25}{23}\right)^{174/529} \\ \Lambda^{(4)} &= \Lambda^{(5)}\left(\frac{m_b}{\Lambda^{(5)}}\right)^{2/25}\left[\ln\left(\frac{m_b^2}{\Lambda^{(5)2}}\right)\right]^{963/14375}\left(\frac{23}{25}\right)^{231/625} \\ \Lambda^{(4)} &= \Lambda^{(3)}\left(\frac{\Lambda^{(3)}}{m_c}\right)^{2/25}\left[\ln\left(\frac{m_c^2}{\Lambda^{(3)2}}\right)\right]^{-107/1875}\left(\frac{27}{25}\right)^{231/625} \\ \Lambda^{(3)} &= \Lambda^{(4)}\left(\frac{m_c}{\Lambda^{(4)}}\right)^{2/27}\left[\ln\left(\frac{m_c^2}{\Lambda^{(4)2}}\right)\right]^{107/2025}\left(\frac{25}{27}\right)^{32/81} \end{aligned} \tag{25}$$

$$\Lambda^{(3)} = \Lambda^{(2)}\left(\frac{\Lambda^{(2)}}{m_s}\right)^{2/27}\left[\ln\left(\frac{m_s^2}{\Lambda^{(2)2}}\right)\right]^{-107/2349}\left(\frac{29}{27}\right)^{32/81}$$
$$\Lambda^{(2)} = \Lambda^{(3)}\left(\frac{m_s}{\Lambda^{(3)}}\right)^{2/29}\left[\ln\left(\frac{m_s^2}{\Lambda^{(3)2}}\right)\right]^{107/2523}\left(\frac{27}{29}\right)^{345/841} \quad (25)$$

These relations differ from those of Marciano (1984) by approximately 3%. From Eqs. (25) we obtain

$$\Lambda^{(4)} = 304(16)\ \text{MeV} \quad (26)$$

$$\Lambda^{(5)} = 216(13)\ \text{MeV} \quad (27)$$

$$\Lambda^{(6)} = 92(6)\ \text{MeV} \quad (28)$$

From equations (22) and (27) we obtain

$$\alpha_s(M_Z) = 0.1186(11) \quad (29)$$

whose error is less than 1%! This is consistent with the experimental value from LEP (Wachsmuth, 1991) and the latest value from SLD (Abe *et al.*, 1993)

$$\alpha_s(M_Z) = \begin{cases} 0.120(7) & \text{LEP} \\ 0.118 \pm 0.002(\text{stat}) \pm 0.003(\text{sys}) \pm 0.010(\text{th}) & \text{SLD} \end{cases} \quad (30)$$

It is clear that a more accurate experimental value is needed. For $\alpha_s(34\ \text{GeV})$ our result is

$$\alpha_s(34\ \text{GeV}) = 0.1396(16) \quad (31)$$

From the experimental value for $R = 3\Sigma Q_f^2 r$,

$$r = 1.049(7) \quad (32)$$

one obtains (Samuel and Surguladze, 1992)

$$\alpha_s(34\ \text{GeV}) = 0.149(21) \quad (33)$$

in agreement with equation (31). These results along with α_s evaluated at 17.3, 31.6, 80.6, and 180 GeV are shown in Table I. It can be seen that all these results are consistent with experiment.

For $N_f = 5$ we have

$$f_{\text{F}}^{(0)} = \left(\frac{\alpha_s}{\pi}\right) + 4.444\left(\frac{\alpha_s}{\pi}\right)^2 + 13.13\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(5)} - 7.929)\left(\frac{\alpha_s}{\pi}\right)^4 \quad (34)$$

and

$$f_{\text{B}}^{(0)} = \left(\frac{\alpha_s}{\pi} + 3.485\left(\frac{\alpha_s}{\pi}\right)^2 + 1.575\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(5)} - 93.14)\left(\frac{\alpha_s}{\pi}\right)^4 \quad (35)$$

Table I. Experimental and Predicted Values for $\alpha_s(\mu)$

μ (GeV)	Theoretical value for $\alpha_s(\mu)$	Experimental value for $\alpha_s(\mu)$
1.7769	0.3233(89)	Input
17.3	0.1591(20)	0.18(5)[a]
31.6	0.1415(16)	0.160(16)[b]
34.0	0.1396(16)	0.148(22)[c]
80.6	0.1209(11)	0.123(25)[d]
91.173	0.1186(11)	0.120(7)[e]
		0.118 ± 0.002 (stat) ± 0.003 (sys) ± 0.010 (th)[f]
180.0	0.1076(9)	—

[a]Akesson *et al.* (1986).
[b]Marshall (1989).
[c]Samuel and Surguladze (1992).
[d]Alitti *et al.* (1991).
[e]Wachsmuth (1991).
[f]Abe *et al.* (1993).

We have neglected the term proportional to $(\Sigma Q_i)^2$, as it should be very small. From equation (34) for $f_{\mathrm{F}}^{(0)}$ we obtain $d_4^{(5)} = 35.2$ and from R_τ we get $d_4^{(5)} = 41.8$. From equation (35) for $f_{\mathrm{B}}^{(0)}$ we get $d_4^{(5)} = 77.8$ and from the series directly $d_4^{(5)} = 6.47$. We shall be conservative and take as our value

$$d_4^{(5)} = 40^{+54}_{-40} \tag{36}$$

The R ratio (Surguladze and Samuel, 1991; Gorishny *et al.*, 1991) in the $\overline{\mathrm{MS}}$ scheme for $N_f = 5$ is

$$R = 3\Sigma Q_f^2\left[1 + \left(\frac{\alpha_s}{\pi}\right) + 1.411\left(\frac{\alpha_s}{\pi}\right)^2 - 12.77\left(\frac{\alpha_s}{\pi}\right)^3 - \frac{1.240(\Sigma Q_f)^2}{3\Sigma Q_f^2}\left(\frac{\alpha_s}{\pi}\right)^3 + R_4\left(\frac{\alpha_s}{\pi}\right)^4\right] = 3\Sigma Q_f^2 r \tag{37}$$

where (Pennington and Ross, 1981)

$$R_4 = d_4^{(5)} - 89.3 \tag{38}$$

and so

$$R_4 = -49^{+54}_{-40} \tag{39}$$

Again we neglect the term proportional to $(\Sigma Q_f)^2$, q, since for the case of interest $q = 1/33$ and this term should be negligible.

Now using our result from

$$\alpha_s(34\ \text{GeV}) = 0.1396(16) \tag{40}$$

we obtain

$$r = 1.0459(5) \tag{41}$$

and

$$R = 3.8350(18) \tag{42}$$

These results are accurate at the 0.05% level!

For 31.6 GeV we obtain

$$\alpha_s(31.6\ \text{GeV}) = 0.1415(16) \tag{43}$$

and, hence,

$$r(31.6\ \text{GeV}) = 1.0465(6) \tag{44}$$

This should be compared to the experimental result (Marshall, 1989)

$$r(31.6\ \text{GeV}) = 1.0527(50) \tag{45}$$

In conclusion, we have shown how one can use our Padé approximant Prediction (PAP) method to estimate R_τ at the five-loop level of PQCD. This estimate has then been used to obtain more accurate predictions for $\alpha_s(\nu)$ for various values of μ. The agreement with experiment is excellent.

We have also used our result for $\alpha_s(34\ \text{GeV})$ to obtain the R ratio accurate to 0.05%. It should be emphasized that once we have fixed $\alpha_s(M_\tau)$ all the results in this paper are determined and have been obtained with no adjustable parameters. Now we need to improve the accuracy of the experimental values!

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